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Exploding Spheres of Dust

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Abstract

In this paper, we extend the work of Oppenheimer & Snyder (1939) who treated the zero curvature case in the gravitational collapse of spheres of dust intended to represent collapsing stars. A solution is given which is valid for all space and is characterized by negative curvature of the space within a sphere of dust. This solution is obtained by matching the negative curvature interior solution (as well as, for completeness, zero and positive curvatures interior solutions) to an exterior Schwarzschild geometry. In this solution, corresponding to the case of a Newtonian system with positive total energy, the mass as seen by an observer at infinity is found to be positive definite. Also, in each case, the positive definite mass m is related to the density ρ and radius r [defined as the square root of the (surface area/4π)] of the dust cloud via $m = (4\pi/3)\rho r^3$. The methods employed here for matching interior and exterior solutions are applicable to the construction of cosmological models in which the sign of the curvature and/or expansion rate differ in two or more regions, e.g. a universe expanding in one region and contracting in another.

1. *Introduction*

In Newtonian mechanics one investigates the expansion or collapse of a homogeneous spherical ball of dust by integrating Newton's equations. The same problem in general relativity is more complicated; it usually involves solving Einstein's field equations in two coordinate patches and then fitting the geometries together. The patching problem is often the most difficult part of the problem. However, in the case of expanding or collapsing dust, an interesting theorem due to Beckedorff & Misner (unpublished observations), and Lindquist & Wheeler (1957) reduces the patching problem almost to the simplicity of Newtonian mechanics. This theorem (Harrison *et al.*, 1965) states that: A necessary and sufficient condition for two 4-geometries to join smoothly (at the interface) is that particles at the interface follow simultaneously the geodesic laws of motion for the two separate geometries. The method is applied by Harrison *et al.* (1965) for the case of positive curvature of the space within the dust. Through the use of this theorem one can match the geometries without explicitly matching the metric tensors and their derivatives.

In this paper we consider the cases of zero and negative curvature as well as (for completeness) positive curvature. One objective is to write the solution in such a way that it can be used in a subsequent paper concerning the gravitational collapse of rotating balls of dust.

In Section 2, Newton's equations for the gravitational collapse of a ball of dust are written down and integrated. This is done to facilitate the interpretation of the relativistic results for the cases of zero and negative curvature. For the case of positive curvature, it is often more convenient to interpret the solution via the initial value equations at the point of time symmetry (Cohen, 1967, 8). In Sections 3 and 4 geodesics are investigated. The geometries are matched and the results are discussed physically in Sections 5 and 6. In Section 7, the effect of rotation is considered.

2. *Newtonian Collapse*

To facilitate the interpretation of later results, a short Newtonian investigation of the expansion or collapse of a dust ball is given here. Newton's equations for the outermost dust particle of the dust ball are

$$M\ddot{r} = -Mmr^{-2} \tag{2.1}$$

Here M denotes the geometrized mass (Harrison *et al.*, 1965) of the dust particle, m the mass of the entire dust ball, r the radius of the ball, and the dot denotes differentiation with respect to time. Integration of equation (2.1) yields

$$\dot{r}^2 = -K_n + 2mr^{-1} \tag{2.2}$$

where $-K_n$ is an integration constant equal to twice the total energy of the particle per unit mass.

If the cases of positive, zero and negative K_n are investigated separately, the solution takes the simple form

$$\begin{aligned} r &= r_0 \sin^2 (\eta/2) \\ t &= (\tfrac{1}{2}) (r_0{}^3/2m)^{1/2} (\eta - \sin\eta) \end{aligned} \tag{2.3a}$$

for $0 < K_n = 2m/r_0$ (def. of r_0),

$$\begin{aligned} r &= \beta t^{2/3} \\ \beta^3 &= 9m/2 \end{aligned} \tag{2.3b}$$

for $K_n = 0$, and

$$r = r_0 \sinh^2 (\eta/2)$$
$$t = (\tfrac{1}{2}) (r_0{}^3/2m)^{1/2} (\sinh \eta - \eta) \tag{2.3c}$$

for $0 > K_n = -2m/r_0$.

For $K_n > 0$ the dust expands and then recontracts. In the other two cases the dust only expands or contracts. In general relativity, the cases of K_n positive, zero and negative correspond to positive, zero and negative curvature of the space within the dust. Also the expressions (2.3a–c) become identical with those of general relativity if the time is replaced by the proper time.

3. *Interior Solution*

The space interior to a homogeneous distribution of dust is often described by the Robertson–Walker metric (Robertson, 1935; Walker, 1937):

$$ds^2 = -dT^2 + a^2(T) [1 + (k/4) U^2]^{-2} [dU^2 + U^2 d\theta^2 + U^2 \sin^2 \theta \, d\varphi^2] \tag{3.1}$$

Here $k = 1, 0, -1$ corresponds to positive, zero or negative curvature of the space within the dust.

The quantity a, which is proportional to the luminosity radius of the dust ball, must satisfy the field equation

$$G^{00} = 8\pi T^{00}$$

In a synchronous frame (Landau & Lifshitz, 1962) on a three-dimensional spacelike surface, this equation takes the form

$$^{(3)}R + K_2 = 16\pi\rho$$

where $^{(3)}R$ is the scalar curvature of the spacelike surface, K_2 is the extrinsic curvature related to the second fundamental form K_{ij} via

$$K_2 = (K_i{}^i)^2 - K^{ij} K_{ij}$$

and ρ is the energy density of the dust.

Direct calculation yields

$$^{(3)}R = 6ka^{-2} \qquad \text{and} \qquad K_2 = 6\dot{a}^2 a^{-2}$$

Here $\dot{a}$ denotes differentiation with respect to T. Consequently the above field equation takes the form

$$ka^{-2} = (8\pi\rho/3) - (\dot{a}/a)^2$$

The Robertson–Walker metric (Robertson, 1935; Walker 1937) is often used to describe a homogeneous and isotropic universe (a Friedman universe). The average density ρ necessary to yield a given Hubble constant $H = \dot{a}/a$ depends on the curvature constant k. A larger density is necessary for positive curvature than for zero or negative curvature as can be seen from inspection of the above equation.

When these three cases are treated separately, we obtain results very similar to those of equation (2.3):

$$\begin{aligned} a &= a_0 \sin^2(\eta/2) \\ T &= (a_0/2)(\eta - \sin\eta) \\ a_0 &= (8\pi/3)\rho a^3 \end{aligned} \tag{3.2a}$$

for $k = 1$,

$$\begin{aligned} a &= \alpha T^{2/3} \\ \alpha^3 &= 6\pi\rho a^3 \end{aligned} \tag{3.2b}$$

for $k = 0$, and

$$\begin{aligned} a &= a_0 \sinh^2(\eta/2) \\ T &= (a_0/2)(\sinh\eta - \eta) \\ a_0 &= (8\pi/3)\rho a^3 \end{aligned} \tag{3.2c}$$

for $k = -1$.

Here ρ is the time dependent density of the dust while a_0 and α are integration constants.

In the space described by this metric there are geodesics for which $U =$ constant and T is the proper time along the geodesic. This is not surprising since the dust was assumed to be co-moving in the derivation of this metric.

4. *Exterior Solution*

The space exterior to a sphere of dust can be described by the Schwarzschild metric:

$$ds^2 = -A^2\,dt^2 + B^2\,dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\varphi^2 \tag{4.1}$$

where

$$A^2 = B^{-2} = 1 - 2mr^{-1}$$

Here the constant m is the mass seen by an observer at infinity. The connection between this mass and the density of the dust can be obtained by matching the interior and exterior solutions (patching).

An especially simple way to obtain the geodesic equations for radial motion of a dust particle in this geometry (Eq. 4.1) is via the variational principle

$$0 = \delta \int ds$$

These geodesic equations are

$$[A^2 t']' = 0 \tag{4.2}$$

and

$$[A^{-2} r']' + (A^2)_r (t')^2 + A^{-4}(r')^2 = 0 \tag{4.3}$$

where the prime denotes differentiation with respect to the proper time $T(dT^2 = -ds^2)$ and $A^2 = 1 - 2mr^{-1}$. Integration of equation (4.2) yields

$$A^2 t' = K \tag{4.4}$$

where K is an integration constant. From equation (4.3), t' can be eliminated via substitution of equation (4.4) yielding

$$[A^{-2}(r')^2]' + K^2 A^{-4}(A^2)' = 0 \tag{4.5}$$

after some manipulation.

Integration of equation (4.5) yields

$$(r')^2 = K^2 + \bar{K}A^2 = K^2 + \bar{K} - 2m\bar{K}r^{-1} \tag{4.6}$$

where $\bar{K}$ is an integration constant. For large r and small velocities, equation (4.6) should reduce to a Newtonian expression (2.2). The two expressions (4.6) and (2.2) agree in this limit if $\bar{K} = -1$. Hence, equation (4.7), like equation (2.2), contains only one adjustable constant K. The first integral (4.6) can also be obtained directly from the line element by dividing all terms by the proper time and eliminating t' via equation (4.4). This latter method gives $\bar{K} = -1$ automatically.

The solution of the general relativistic equations of motion takes a simple form when the three cases $(1 - K^2)$ positive, zero and negative are treated separately. The solution is very similar to that for the Newtonian case:

$$r = r_0 \sin^2(\eta/2)$$

$$T = (\tfrac{1}{2})\,(r_0{}^3/2m)^{1/2}\,(\eta - \sin\eta) \tag{4.7a}$$

for $0 < 1 - K^2 = 2mr_0^{-1}$,

$$r = \beta T^{2/3}$$

$$\beta^3 = 9m/2 \tag{4.7b}$$

for $K^2 = 1$, and

$$r = r_0 \sinh^2(\eta/2)$$
$$T = (\tfrac{1}{2})(r_0{}^3/2m)^{1/2}(\sinh\eta - \eta) \tag{4.7c}$$

for $0 > 1 - K^2 = -2mr_0^{-1}$.

The expressions† (2.3a–c) and (4.7a–c) differ only in that the time in equation (2.3a–c) is replaced by the proper time in equation (4.7a–c).

The solution (4.7a–c) describes the motion of a test particle in gravitational field of a spherically symmetric body of mass m. When the massive body is composed of gravitationally collapsing dust, there is a connection between the mass seen by an observer at infinity and the density of the dust. This connection is obtained by matching the interior and exterior solutions.

5. *Matching of Geometries and Discussion*

Application of the theorem discussed in Section 1, reduces the problem of matching 4-geometries to that of matching geodesics. In Sections 3 and 4 the *geodesics can be matched by inspection.* The motion of the same particle is describable in two different coordinate patches. Thus, in each description, the proper time of a particle at the interface between the two geometries and the proper circumference of the ball of dust should agree.

In view of the above discussion, comparison of equations (3.2a–c) and (4.7a–c) yields

$$[1 + (k/4)\,U_0{}^2]^{-1} a_0 U_0 = r_0$$
$$a_0 = (r_0{}^3/2m)^{1/2} \tag{5.1a}$$

for $k = 1$ or -1, and

$$\alpha U_0 = \beta \tag{5.1b}$$

for $k = 0$.

Thus, if the conditions (5.1a–b) are satisfied, the 4-geometries will join smoothly at the interface. These conditions (5.1a–b) when combined with equations (3.2a–c) and (4.7a–c) yield‡

$$m = (4\pi/3)\,\rho r^3 \tag{5.2}$$

† The origin and sign of the time in the above expression are arbitrary. Thus the above solutions are valid if the time t is replaced by $\pm(t - t_0)$. This is also true of the previous solutions. The substitution $t \to -t$ gives an initially collapsing solution.

‡ For the case where $k = 1$ or -1 this result is obtained as follows:
$1 = (8\pi/3)\,\rho a^3/a_0 = (8\pi/3)\,\rho r^3(1 + (k/4)\,U_0{}^2)^3/a_0 U_0{}^3 = (8\pi/3)\,\rho r^3 a_0{}^2 r_0^{-3}$
$= (4\pi/3)\,\rho r^3/m.$

for each case. Hence, the general relativistic expression for the mass of a homogeneous ball of dust is identical with the well known Newtonian expression in terms of the density ρ and radius r of the dust ball. Also, in each case the proper time required for collapse from a finite radius to the gravitational radius is finite.

To an observer at infinity, the radius of the dust ball always appears to be greater than the Schwarzschild radius. Thus, an observer is unable to see the latter stages of the collapse because of the time dilation. Since no energy escapes from the dust ball, the mass seen by this observer is constant. On the other hand, if the ball of dust (with $r > 2m$) is initially expanding it will recontract only for the case $k = 1$.

Using the method described above there can be constructed a universe which has sections with negative as well as positive curvature and with all of these sections fitting together smoothly. A simple example can be constructed as follows: In the above calculations we have obtained the geometry exterior and interior to a collapsing dust ball with negative curvature within it. If a positive curvature Friedman universe is then patched to the large r portion of the Schwarzschild region, we have a universe. The method used to patch these two regions together is identical with that used in this paper to patch the dust ball to the Schwarzschild region. It is not known whether the resulting universe is open or closed.

As an aid to visualizing the situation, consider a closed universe at the instant of time symmetry containing a Schwarzschild region with a positive curvature Friedman region at each end. Such a universe is shown in Fig. 1.

The universe with a negative curvature section is very similar, it has a Schwarzschild region with a positive curvature Friedman region on one end and a negative curvature Friedman region on the other. Such a universe can expand in one region and contract in another.

Other results of physical interest can be obtained via differentiation of equation (4.7a–c) with respect to the proper time, or substitution of equation (4.7a–c) into equation (4.6), yielding

$$r' = (2m/r)^{1/2}(1 - rr_0^{-1})^{1/2} \tag{5.3a}$$

for $k = 1$,

$$r' = (2m/r)^{1/2} \tag{5.3b}$$

for $k = 0$, and

$$r' = (2m/r)^{1/2}(1 + rr_0^{-1})^{1/2} \tag{5.3}$$

for $k = -1$.

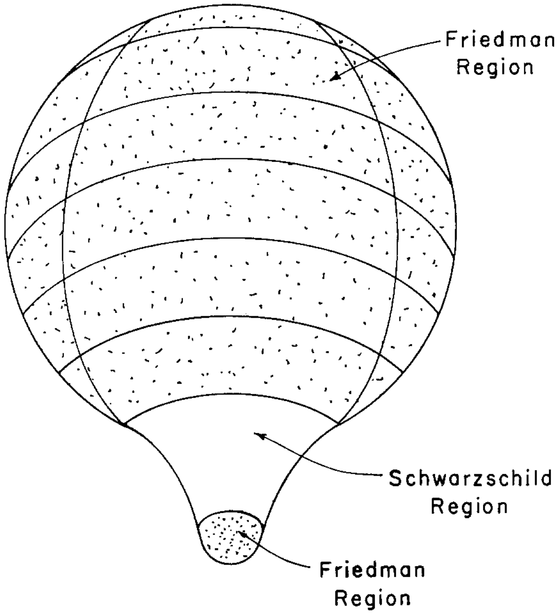

Figure 1.—Closed inhomogeneous universe at the moment of time symmetry. This universe is obtained by patching positive curvature Friedman regions to both ends of a Schwarzschild region.

This velocity r' is related to the velocity $\dot{r} = dr/dt$ via the line element (Eq. 4.1) $\dot{r}^2 = A^4(r')^2[A^2 + (r')^2]^{-1}$. Thus, the velocity seen by an observer at infinity (who remains at a constant distance from the center of the dust ball) is

$$\dot{r} = A^2[2mr^{-1} - 2mr_0^{-1}]^{1/2}[1 - 2mr_0^{-1}]^{-1/2} \tag{5.4a}$$

for $k = 1$,

$$\dot{r} = A^2(2mr^{-1})^{1/2} \tag{5.4b}$$

for $k = 0$, and

$$\dot{r} = A^2[2mr^{-1} + 2mr_0^{-1}]^{1/2}[1 + 2mr_0^{-1}]^{1/2} \tag{5.4c}$$

for $k = -1$.

In Newtonian mechanics, there are three cases: (1) the dust expands and then recontracts, (2) the dust expands continuously with the expansion velocity $\dot{r}$ approaching zero as r approaches infinity, and (3) the dust expands continuously but the expansion velocity approaches a finite value as r approaches infinity. Inspection of equation (5.4a–c) shows that in general relativity these three cases correspond to positive,† zero, and negative curvature respectively. In the latter case $\dot{r}$ approaches $(2mr_0^{-1})^{1/2}(1+r_0^{-1}2m)^{1/2}$ as r approaches infinity.

When combined with equation (5.4a–c), equation (4.4) has a simple physical interpretation. For $k=1$ and $r=r_0$, equation (4.4) takes the form

$$dT = (1-2mr_0^{-1})^{1/2}\,dt \tag{5.5}$$

which is the well-known relation between time intervals of the Schwarzschild solution. For $k=0$ and $k=-1$, the dust expands to infinite radius. Consequently, the expression‡ can be compared with the results of special relativity, since the Schwarzschild geometry is asymptotically flat. According to special relativity, the time interval dT measured by an observer sitting on a moving particle is shorter than the time interval dt seem by a stationary observer, i.e.

$$dT = (1-\dot{r}^2)^{1/2}\,dt \tag{5.6}$$

Substitution of the value of $\dot{r}$ as r approaches infinity yields

$$t' = (dT/dt) = 1 \tag{5.7}$$

for $k=0$, and

$$t' = (1+2mr_0^{-1})^{1/2} \tag{5.8}$$

for $k=-1$ where $t' = dT/dt$.

These expressions are identical with equation (4.4) for $r \to \infty$ and $k=0$ and -1 respectively. Thus, in certain regions the results of equation (4.4) can be obtained by simple physical arguments.

6. *Discussion of Astrophysical Applications*

These results allow a general relativistic description of idealized astrophysical systems which expand and never recontract as well as those which recontract, e.g. exploding stars (supernovae), galaxies, or clusters of galaxies at the stage where the pressure and rotation are negligible. The negative curvature case corresponds to an unbound

† For a discussion of a ball of dust with positive curvature via the initial value equations, see Harrison *et al.* (1965). An extensive bibliography of previous work on the subject is also given there.

‡ A similar problem was considered by Wahlquist & Estabrook (1967). *Physical Review*, **156**, 1359; see also, e.g., Oppenheimer & Snyder (1939). *Physical Review*, **56**, 455; McVitte (1964), *Astrophysical Journal*, **140**, 401.

system in Newtonian mechanics, one which never recontracts if it is initially expanding. The zero curvature case corresponds to the transition between bound and unbound systems (the zero total energy case); it just barely expands to infinity using up all its kinetic energy in the process. Although it is possible, it is very unlikely in the $k = -1$ case that the velocity vectors will be arranged such that the system will collapse. Consequently, in a physical situation, the $k = -1$ case is more likely to be expanding than contracting.

Using the method discussed in this paper, one can construct general relativistic supernovae models. The expanding envelope of the supernovae corresponds to the expanding negative curvature solution given here. For example, to construct such a model of a supernova, patch an interior solution (collapsing or static) to an exterior Schwarzschild solution, then patch this to an expanding negative curvature solution, and finally patch this to another exterior Schwarzschild solution. For such a model, the envelope blows off, leaving a remnant behind, in agreement with non-relativistic models. The mass of this remnant m_r is given by

$$m_r = (4\pi/3)\,\rho {r_1}^3$$

where ρ is the density and r_1 is the inner radius of the envelope. In this way, we can treat the expansion or collapse of thick shells of dust.

7. *Effect of Rotation*

An important question in astrophysics is: Does rotation stop collapse or does collapse crush rotation? (Hoyle *et al.*, 1964). In order to study this question, consider a test particle on the surface of the collapsing ball of dust. In Newtonian mechanics one learns that if the particle has sufficient tangential velocity, it will remain in orbit around the collapsing dust. If the particle remains in orbit, it has a point of closest approach to the center of the collapsing dust ball. At this point the outward radial acceleration is equal to or greater than zero. The former case corresponds to circular motion.

The general relativistic equations of motion of the test particle can be obtained from the variational principle (4.1) yielding

$$(r^2\varphi')' = 0 \tag{7.1a}$$

$$(A^2 t')' = 0 \tag{7.1b}$$

$$2B^2 r'' + (B^2)_r (r') = 2r(\varphi')^2 - (A^2)_r (t')^2 \tag{7.1c}$$

Integration of equations (7.1a) and (7.1b) yields

$$r^2 \varphi' = l \tag{7.2a}$$

$$A^2 t' = K \tag{7.2b}$$

and substitution into equation (7.1c) yields

$$2B^2 r'' + (B^2)_r (r')^2 = 2l^2 r^{-3} - 2mr^{-2} K^2 A^{-4} \tag{7.2c}$$

At the test particle's point of closest approach to the center of the dust ball, the radial velocity r' vanishes and the radial acceleration r'' is equal or greater than zero. This implies that

$$l^2 \geqslant mrK^2 A^{-4} \tag{7.3}$$

On the other hand, the world line of the particle must remain within the light cone. This is true only if

$$l^2 \leqslant K^2 r^2 A^{-2} \tag{7.4}$$

at the point of closest approach. These two conditions (7.3) and (7.4) are consistent only for $r \geqslant 3m$.

Thus, for $r \geqslant 3m$, rotation of the test particle about the center of the dust ball can stop the collapse of the particle.† The particles can remain in orbit around the collapsing dust ball. However, for $3m > r > 2m$ the gravitational attraction overpowers the centrifugal force. Thus, in this region the rotation cannot stop the collapse. This is because the expression for centrifugal force deviated from the Newtonian one since the particle world line must remain within the light cone; the local velocity of the particle cannot exceed that of light.

A collapsing star can shed some of its angular momentum via the above mechanism and collapse towards its Schwarzschild radius leaving planets in orbit about it. But for $3m > r > 2m$ the planets cannot remain in orbit. The centrifugal force is overpowered by the gravitational attraction and everything collapses.

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† For a complete discussion of such orbits see Hagihara, Y. (1931). *Japanese Journal of Astronomy and Geophysics*, **VIII**, 67. See also, for example, Kuchowicz, B. (1966). *Acta physiologica polonica*, **30**, 981, and the references cited there.

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